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LETTER TO THE EDITOR

Dressing transformation for a many atom–radiation field Hamiltonian

Lorenzo M Narducci† and Charles M Bowden

Quantum Physics, Physical Sciences Directorate, Redstone Arsenal, AL 35809, USA

Abstract. We apply Coulter's unitary transformation to the Hamiltonian for N two-level systems interacting with the radiation field. The resulting dressed Hamiltonian displays single-atom and collective frequency renormalization terms, as well as retarded dipole–dipole interaction contributions in a very direct way. The existence of field-intensity-dependent frequency shifts is also discussed.

A serious conceptual difficulty with quantum electrodynamics is generated by our inability to deal with physical rather than bare (or unperturbed) states of particles interacting with the radiation field. It is true that many radiative processes can be described quite adequately under the assumption that the interaction Hamiltonian induces transitions between bare states. This is, of course, only an approximation, as our experience with quantum field theories indicates that field interactions are responsible not only for transitions between states but also for essential modifications of the states themselves (Bogoliubov and Shirkov 1959). This situation, which can be remedied in part by appropriate renormalization processes (Heitler 1957a), stems from the absence of a clear prescription for identifying the physical (dressed) states of interacting systems.

Recently Coulter (1974) suggested a canonical transformation that attemps to remove the difficulty associated with the identification of the physical states of the system. His transformation, similar in spirit to the one proposed by Heitler (1957b) for time-dependent calculations, maps the (unknown) eigenstates of a total Hamiltonian into a new set of basis vectors in such a way that the dressed state of lowest energy is formally the same as the bare ground state of the original basis. As a result of a careful analysis of the transformed Hamiltonian, Coulter suggested that *also* the excited dressed states in the new basis are represented by the appropriate bare state vectors of the old basis. Thus, after application of the unitary transformation, the states $|+\rangle|\text{vac}\rangle$, $|-\rangle a_k^\dagger|\text{vac}\rangle$, etc of a two-level atom in a radiation field would represent the *physical*, excited state in vacuum, the atomic ground state with one photon present in mode k, etc. Transitions between the physical states are now induced by the transformed, or dressed, Hamiltonian.

The determination of the unitary transformation is based on a straightforward, although lengthy, perturbation procedure, and it requires only algebraic manipulations. The central problem, in our opinion, rests with the identification of the dressed quantum states. Since, however, Coulter's conjecture looks very appealing judging

† Present address: Drexel University, Philadelphia 19104, USA.

from the applications discussed in his paper, we have decided to re-examine some aspects of the problem of the interaction of many atoms with radiation using his dressing transformation theory. The results to be discussed in this letter provide support to Coulter's conclusions. We show that the transformed multi-atom Hamiltonian exhibits delayed dipole–dipole interaction contributions (the so called Stephen potential (Stephen 1964, Lemberg 1970, Dillard and Robl 1969, Agarwal 1973, Banfi and Bonifacio 1974, 1975, Stroud *et al* 1972, Morawitz 1973, Friedberg and Hartmann 1974)) in a very natural way. In addition, we identify the well known single-atom frequency renormalization term and predict a collective intensity-dependent frequency shift which should provide non-negligible contributions if the initial field state is not a state of vacuum.

Our calculation takes its starting point from the popular model of N two-level atoms interacting with one another only through the electromagnetic field. Following Banfi and Bonifacio (1974, 1975), the atoms are described in terms of collective operators defined by

$$R_\alpha^\pm = \sum_{i=1}^{N} r_i^\pm \, e^{\pm i\boldsymbol{\alpha}\cdot\boldsymbol{x}_i}, \qquad R_{3\alpha} = \sum_{i=1}^{N} r_{3i} \, e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}_i} \qquad (1)$$

where $\boldsymbol{x}_i$ denotes the position of the ith atom on a cubic lattice (lattice constant d) and $\boldsymbol{\alpha}$ labels the atomic collective modes in the volume V_c of the system ($\alpha_i = (2\pi/L_i)n_i$, n_i integer, $-\pi d^{-1} \le \alpha_i \le \pi d^{-1}$, $i = x, y, z$). The field is quantized over a large volume $V \gg V_c$ in the Coulomb gauge. At the end of the calculation an appropriate limiting process will be defined by which V will be extended to infinity.

Neglecting electrostatic dipole interactions, the Hamiltonian takes the form

$$H = \sum_k \omega_k a_k^\dagger a_k + \Omega R_3 + \sum_{k,\alpha} g_k (a_k R_\alpha^+ f(k - \boldsymbol{\alpha}) + a_k^\dagger R_\alpha^- f^*(k - \boldsymbol{\alpha}))$$

$$+ \sum_{k,\alpha} g_k (a_k^\dagger R_\alpha^+ f^*(k + \boldsymbol{\alpha}) + a_k R_\alpha^- f(k + \boldsymbol{\alpha})) \qquad (2)$$

where, in the dipole approximation,

$$g_k = \Omega(2\omega_k)^{-1/2}[\mu_{12}^2 - (\hat{k}\cdot\boldsymbol{\mu}_{12})^2]^{1/2}$$

is the coupling constant and $\boldsymbol{\mu}_{12}$ is the electric dipole matrix element. In the optical domain and with a suitable choice of reference systems the 'diffraction functions' $f(k \pm \boldsymbol{\alpha})$ can be taken to be real and independent of the lattice structure (Banfi and Bonifacio 1974, 1975):

$$f(k \pm \boldsymbol{\alpha}) = \prod_{i=1}^{3} \mathrm{sinc}(\tfrac{1}{2}(k \pm \boldsymbol{\alpha})_i L_i). \qquad (3)$$

The Coulter unitary transformation is applied in a perturbative way and is carried out to second order in the coupling constant g_k. This level of approximation is satisfactory as it enables us to compare our results with most of the published work, which is usually based on the first Born approximation. It should be noticed, however, that collective atomic decay has been discussed, using an exact master equation, by Bonifacio and Lugiato (1975a,b, see also Gronchi and Lugiato 1976). The details of the calculation are quite simple, but algebraically involved, and will be discussed in a separate publication. Here we merely quote the final result: *The dressed Hamiltonian, which is unitarily equivalent to the Hamiltonian (2) up to second order in the coupling*

constant g_k, *is given by the sum of three terms* H_0, D_1, D_2. *The term* H_0, *given by*

$$H_0 = \sum_k \omega_k a_k^\dagger a_k + \Omega R_3, \tag{4}$$

is the unperturbed Hamiltonian. The dressed interaction terms are given by

$$D_1 = \sum_{l \in \delta\Omega} \sum_\alpha g_l f(l - \alpha)(a_l R_\alpha^+ + a_l^\dagger R_\alpha^-) \tag{5}$$

and

$$D_2 = \sum_{k \neq l} \sum_{\alpha,\beta} \frac{g_k^2}{\Omega - \omega_k} f(k - \alpha) f(k - \beta) R_\alpha^+ R_\beta^- - \sum_k \sum_{\alpha,\beta} \frac{g_k^2}{\Omega + \omega_k} f(k + \alpha) f(k + \beta) R_\beta^- R_\alpha^+$$

$$+ 2 \sum_{k \neq l} \sum_{\alpha,\beta} \frac{g_k^2}{\Omega - \omega_k} f(k - \alpha) f(k - \beta) a_k^\dagger a_k R_{3,\alpha-\beta}$$

$$+ 2 \sum_k \sum_{\alpha,\beta} \frac{g_k^2}{\Omega + \omega_k} f(k + \alpha) f(k + \beta) a_k^\dagger a_k R_{3,\alpha-\beta}. \tag{6}$$

The mode summation in equation (5) is restricted over a thin shell of k values (labelled $l \in \delta\Omega$) such that $|k| \approx \Omega/c$. In the limit as the quantization volume approaches infinity, it is required that the thickness of the k shell be made vanishingly small (Heitler 1957c). The summations labelled $k \neq l$ in equation (6) range over all posible values of k except for those in the above mentioned shell of k space. (In practice, in the infinite-volume limit these restricted summations become principal part integrals $P\int(d^3k/(2\pi)^3]$.)

We notice at once that D_1 describes energy exchanges between the atoms and the field. In the infinite-volume limit, as the mode distribution approaches a continuum, D_1 becomes responsible for the irreversible energy transfer from the atoms to the field (see below). It is also interesting that D_1 does not contain counter-rotating terms. It is well known (Ackerhalt *et al* 1973a, Ackerhalt *et al* 1973b, Senitsky 1972) that these terms are responsible for part of the atomic frequency renomalization; their effect is accounted for in D_2 to order g_k^2.

The operator D_2 is comprised of two separate contributions. The last two terms are responsible for intensity-dependent level shifts. However, they do not contribute (to order g_k^2) if the initial field state is the state of vacuum (see below). The first two terms contain single-atom and collective frequency renormalization contributions. Specializing to $\alpha = \beta$, for example, the vacuum contribution $D_2^{(0)}$ takes the form

$$D_2^{(0)}(\alpha = \beta) = \tfrac{1}{2} N \mathcal{V} + \sum_{i \neq j} r_i^+ r_j^- \mathcal{V}_{ij} + \Omega' R_3 \tag{7}$$

where

$$\mathcal{V}_{ij} = \sum_\alpha \left(\sum_{k \neq l} \frac{g_k^2}{\Omega - \omega_k} f^2(k - \alpha) - \sum_k \frac{g_k^2}{\Omega + \omega_k} f^2(k + \alpha) \right) e^{i\alpha \cdot (x_i - x_j)}$$

$$\Omega' = \sum_\alpha \left(\sum_{k \neq l} \frac{g_k^2}{\Omega - \omega_k} f^2(k - \alpha) + \sum_k \frac{g_k^2}{\Omega + \omega_k} f^2(k + \alpha) \right) \tag{8}$$

and $\mathcal{V} \equiv \mathcal{V}_{ii}$. The term in equation (7) describing spin–spin interaction, and usually referred to as the retarded dipole–dipole interaction, is responsible for collective

frequency shifts, while $\Omega' R_3$ is responsible for the single-atom frequency renormalization.

All of the above contributions have been derived and discussed in many previous publications usually in the context of a time-dependent analysis of the multi-atom emission process. From our calculation, it becomes obvious that these terms can also be regarded as inherent features of the Hamiltonian itself provided we analyse our system from the new dressed unitary frame.

The terms in D_2 corresponding to $\alpha \neq \beta$ are responsible for atomic mode–mode coupling effects. Their importance to the dynamical evolution has been the subject of rather active debate. This issue has been carefully analysed by Banfi and Bonifacio (1974, 1975) with the conclusion that the $\alpha \neq \beta$ terms are dynamically unimportant for pencil-shaped geometries with a Fresnel number of the order of or greater than unity. Qualitatively, from equation (6) one is tempted to arrive at the same conclusion because the sharply peaked nature of the diffraction functions should cause the products $f(k \pm \alpha) f(k \pm \beta)$ to provide an approximate selection rule $\alpha \approx \beta$. This conjecture, however, will have to be analysed in greater detail.

Our physical interpretation of the effects of the dressed operators D_1 and D_2 can be made more explicit if we construct from the dressed Hamiltonian the master equation for the atomic density operator $W_A(t)$. If the initial field state is the vacuum, the master equation in the interaction picture and in the first Born approximation takes the form

$$\dot{W}_A(t) = -i \, \text{Tr}_F \mathcal{L}_2(|0\rangle\langle 0| W_A(t)) - \int_0^t d\tau \, \text{Tr}_F \mathcal{L}_1(t) \mathcal{L}_1(\tau)(|0\rangle\langle 0| W_A(\tau)) \quad (9)$$

where Tr_F represents the trace over the field variables and the Liouvillian operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as

$$\mathcal{L}_2 = [D_2, \ldots], \qquad \mathcal{L}_1(t) = [D_1(t), \ldots]. \quad (10)$$

A brief outline of the procedure is given by Banfi and Bonifacio (1975, appendix A). A more detailed treatment of projector techniques is reviewed by Haake (1973).

In the Markoff approximation and carrying out the indicated trace operation over the field variables, equation (9) takes the form

$$\dot{W}_A(t) = -i[D_2^{(0)}, W_A(t)] + \tfrac{1}{2} \sum_{\alpha,\beta} \gamma_{\alpha\beta}^-([R_\beta^-, W_A R_\alpha^+] + [R_\beta^- W_A, R_\alpha^+]) \quad (11)$$

where

$$\gamma_{\alpha\beta}^- = 2 \sum_{l \in \delta\Omega} g_l^2 f(l - \alpha) f(l - \beta) \pi \delta(\Omega - \omega_l). \quad (12)$$

Thus we confirm that in the infinite-volume limit, the thickness of the k shell can be made to vanish. (In the Markoff approximation, this is accomplished automatically by the δ function in equation (12).) From equations (9) and (11), we are able to restate that the dressed operator D_1 is responsible for the irreversible energy transfer from the atoms to the field, while $D_2^{(0)}$ contains contributions to the frequency renormalization of both single and collective atomic origin.

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